# THE LOBACHEVSKY SPACES IN THE NON-COMMUTATIVE MINKOVSKY SPACES

M. A. Olshanetsky, V.-B. K. Rogov

*Institute Theor. Exper. Physics, Moscow, Russia*
*Moscow State University of Communications (MIIT), Russia*

The final goal of this work is the solutions of the Klein-Gordon equations on NCMS in terms of the horospheric coordinates [1], [2]. By analogy with the classical case, the solutions are products of $q$-cylindric functions. The reduction of these solutions to NCLS, NCILS and the non-commutative cone is straightforward.

*Notations.*
Classical variables are denoted by small letters, while their non-commutative deformation (quantization) by capital letters. We do not introduce a special notation for the non-commutative multiplication.

The coordinates $(x_1, x_2, x_3, x_4)$ or $(y_0, y_1, y_2, y_3)$ of the Minkowski space $\mathbf{M}^4$ we identify with the matrix elements of the matrix $\mathbf{x}$

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ &= y_0 Id + \sum_\alpha \epsilon_\alpha y_\alpha \sigma_\alpha\,, \quad \epsilon_\alpha = 1\,, \text{ or } i\,. \end{aligned} \tag{1}$$

The choice 1 or $i$ in front of $y_\alpha$ defines the signature of the Minkowski space. The generators of the non-commutative Minkowski space we also arrange in the matrix form

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}. \tag{2}$$

The deformation parameter is $q = \exp\theta \in (0, 1]$, or $q = \exp i\theta$, $(|q| = 1)$.

## 1 Horospheric coordinates on the classical Minkowski spaces

There are two types of Minkowski spaces with the signature $(+,-,-,-)$ and $(+,+,-,-)$. The first one allows us to describe the Lobachevsky space $\mathbf{L}$ and the Imaginary Lobachevsky space $\mathbf{IL}$. We will consider first of them.

### 1.1 Minkowski space in the horospheric description

The Minkowski space $\mathbb{M}^{1,3}$ can be identified with the space of Hermitian matrices

$$\mathbb{M}^{1,3} = \{\mathbf{x} \in \mathrm{Mat}_C \mid \mathbf{x}^\dagger = \mathbf{x}\}\,, \quad (\bar{x}_1 = x_1\,,\ \bar{x}_2 = x_3\,,\ \bar{x}_4 = x_4)\,.$$

The metric is $ds^2 = det(d\mathbf{x}) = dx_1dx_4 - dx_2dx_3$. Another set of coordinates is $y_a$, $(a = 0, \dots 3)$ corresponds to the choice $\epsilon_a = 1$ in (1)

$$\mathbf{x} = \sum_{a=0}^{3} y_a\sigma_a\,, \quad \sigma_0 = Id\,.$$

It leads to the metric

$$ds^2 = dy_0^2 - \sum_{j=1}^{3} dy_j^2\,.$$

The group SL(2, $\mathbb{C}$) is the double covering of the proper Lorentz group $SO_0(1,3)$ and acts on the Minkowski space $\mathbb{M}^{1,3} = \{y_0, \dots, y_3\}$ as

$$\mathbf{x} \to g^\dagger \mathbf{x} g\,, \quad g \in \mathrm{SL}(2, \mathbb{C})\,, \tag{3}$$

where $g^\dagger$ is the Hermitian conjugated matrix. The action preserves

$$\det \mathbf{x} = y_0^2 - y_1^2 - y_2^2 - y_3^2 = x_1x_4 - x_3x_2$$

and thereby the metric on $\mathbb{M}^{1,3}$.

The time-like part $\mathbb{M}^{1,3+}$ of $\mathbb{M}^{1,3}$ corresponds to the matrices with $\det \mathbf{x} > 0$, while $\det \mathbf{x} < 0$ corresponds to the space-like part $\mathbb{M}^{1,3-}$. The equation $\det \mathbf{x} = 0$ selects the light cone

$$\mathbf{C}^{1,3} = \{\mathbf{x} : \det \mathbf{x} = x_1x_4 - x_2x_3 = 0\}\,. \tag{4}$$

We introduce *the horospheric coordinates* $\mathbf{x} \sim (r, h, z, \bar{z})$. If $x_1 \neq 0$ then

$$x_1 = rh\,, \quad x_2 = rhz\,, \quad x_3 = rh\bar{z}\,, \tag{5}$$

$$x_4 = r(h|z|^2 + \varepsilon h^{-1})\,.$$

Here

$$z \in \mathbb{C}\,, \quad h \in R \setminus 0\,, \ \varepsilon = \pm 1, 0\,, \ r^2\varepsilon = \det \mathbf{x}\,.$$

and

$$z = x_2x_1^{-1}\,, \ \bar{z} = x_3x_1^{-1}\,, \ r = \sqrt{|\det \mathbf{x}|}\,, \text{ for } \det \mathbf{x} \neq 0\,,$$

$$h = \begin{cases} x_1(|\det \mathbf{x}|)^{-1/2} & \text{for} \quad \varepsilon = \pm 1\,, \\ x_1 & \text{for} \quad \varepsilon = 0\,. \end{cases}$$

The case $\varepsilon = 1$ corresponds to the time-like part of $\mathbb{M}^{1,3}$, $\varepsilon = -1$ corresponds to the space-like part and $\varepsilon = 0$ to the light-cone $\mathbf{C}^{1,3}$.

The horospheric coordinates on the light-cone $\mathbf{C}^{1,3}$ are $(h, z, \bar{z})$

$$x_1 = h\,, \ x_2 = hz\,, \ x_3 = \bar{x_2}\,, \ x_4 = h|z|^2\,. \tag{6}$$

To describe the case $x_1 = 0$ we put $\varepsilon = -1$, $h \to 0$, $r < \infty$ and $z \to \infty$ such that $\lim hz = \exp(it)$, $x_2 = r\exp(it)$ and $x_4$ takes an arbitrary real value. Thus, the horospheric description has the form

$$(r, \exp(it), x_4)\,, \quad x_2 = r\exp(it) \ \ x_3 = r\exp(-it).$$

Consider the commutative algebra $\mathcal{S}(\mathbb{M}^{1,3})$ of the Schwartz functions on $\mathbb{M}^{1,3}$. The invariant integral with respect to the SL(2, $\mathbb{C}$) action on $\mathcal{S}(\mathbb{M}^{1,3})$

$$I(f) = \int f(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4$$

takes the form in the horospheric coordinates

$$I(g) = \int g(z, \bar{z}, h, r) r^3 h dr dh dz d\bar{z}\,, \quad g \in \mathcal{S}(\mathbb{M}^{1,3})\,.$$

## 1.2 Homogeneous spaces, embedded in $\mathbb{M}^{1,3}$

The action of SL(2, $\mathbb{C}$) (3) leads to the foliation of $\mathbb{M}^{1,3}$. The orbits are defined by fixing $\det \mathbf{x}$. The quadric

$$\mathbf{L} = \{\det \mathbf{x} = r_0^2 > 0\,, \quad x_1 > 0\}$$

is the upper sheet of the two-sheeted hyperboloid. It is a model of the Lobachevsky space. The metric on $\mathbf{L}$ is the restriction of the invariant metric $dx_1 dx_4 - dx_2 dx_3$ on $r$ = const. In what follows we assume $r_0 = 1$. The horospheric coordinates on $\mathbf{L}$ have the restrictions $h > 0$. Since SU(2) leaves the point $y_0 = 1$, $y_\alpha = 0$ the Lobachevsky space is the coset $\mathbf{L} \sim \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) \sim \mathrm{SO}_0(1,3)/\mathrm{SO}(3)$.

Consider the commutative algebra $\mathcal{S}(\mathbf{L})$ of Schwartz functions on $\mathbf{L}$. Functions from $\mathcal{S}(\mathbf{L})$ are infinitely differentiable with all derivatives tending to zero when $|z| \to \infty$, $h \to \infty$, $h \to 0$ faster than any power. Let $I_{r_0^2}$ be the ideal in $\mathcal{S}(\mathbb{M}^{1,3})$ generated by $f(\det \mathbf{x} - r_0^2) = 0$. The algebra $\mathcal{S}(\mathbf{L})$ can be described as the factor-algebra $\mathcal{S}(\mathbb{M}^{1,3})/I_1$ with the additional condition $x_1 > 0$. In the similar way we describe the upper sheet of the light-cone $\mathbf{C}^{1,3}$ as $\mathcal{S}(\mathbb{M}^{1,3})/I_0$. The horospheric coordinates (5) being restricted on $\mathbf{C}^{1,3+}$ satisfy the condition $(r = 1\,,\ h > 0\,,\ \varepsilon = 0)$. $\mathbf{C}^{1,3+}$ is the quotient $\mathrm{SL}(2,\mathbb{C})/B_{\mathbb{C}}$, where $\tilde{B}_{\mathbb{C}}$ is the subgroup of the form

$$\tilde{B}_{\mathbb{C}} = \left\{ \begin{pmatrix} \exp(i\phi) & w \\ 0 & \exp(-i\phi) \end{pmatrix}\,,\ w \in \mathbb{C} \right\}.$$

The space

$$\mathbf{IL} = \{\det \mathbf{x} = -1\}$$

is called the *Imaginary Lobachevsky space.* The corresponding quadric is $y_0^2 - \sum_\alpha y_\alpha^2 = -1$. It is the de Sitter space:

$$\mathbf{IL} \sim \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(1,1) \sim \mathrm{SO}_0(1,3)/\mathrm{SO}_0(1,2),$$

since

$$g^\dagger \sigma_3 g = \sigma_3\,, \quad \text{for } g \in \mathrm{SU}(1,1)\,.$$

As before, $\mathcal{S}(\mathbf{IL}) \sim \mathcal{S}(\mathbb{M}^{1,3})/I_{-1}$, but in contrast with the $\mathbf{L}$ and $\mathbf{C}^+$ the horospheric radius $h$ of $\mathbf{IL}$ can take an arbitrary value $h \in \mathbb{R} \setminus 0$. We partially compactify $\mathbf{IL}$ with respect to the coordinate $h$. Two "limiting" spaces $\Xi^\pm = \{h \to \pm\infty\}$ are called *absolutes.* It follows from (4) and (6) that $\Xi^\pm$ can be considered as the projectivization of the cone $\mathbf{C}^{1,3}$. The both absolutes are homeomorphic to $\mathbb{C}$ and therefore can be compactify to $\overline{\Xi}^\pm \sim \mathbb{C}P^1$. Note, that while $\overline{\Xi}^\pm$ are two components of the boundary of the $\mathbf{IL}$, $\overline{\Xi}^+$ is the boundary of $\mathbf{C}^{1,3+}$ and the $\mathbf{L}$.

## 2 Laplace operator and its eigenfunctions

In this work we generalize to the non-commutative case the following facts concerning the eigenfunctions of the Laplace operator.

The solutions of the Klein-Gordon equation on $\mathbb{M}^4$

$$\Delta f_\nu(x_1,x_2,x_3,x_4) = \nu^2 f_\nu(x_1,x_2,x_3,x_4)\,,$$

$$\Delta = \frac{\partial^2}{\partial x_1 \partial x_4} - \frac{\partial^2}{\partial x_3 \partial x_2}\,.$$

are the exponents

$$f_\nu(x_1,x_2,x_3,x_4) = \exp(\xi x)\,,\ (\xi x) = \sum \xi_i x_i\,,$$

$$\nu^2 = \xi_1\xi_4 - \xi_2\xi_3\,.$$

We will consider $\Delta$ and its eigenfunctions in the horospheric coordinates. The metric on $\mathbb{M}^{1,3}$ in the horospheric coordinates takes the form

$$ds^2 = g_{jk}dx_j dx_k = \varepsilon dr^2 - \varepsilon r^2 h^{-2} dh^2 - r^2 h^2 dz d\bar{z}\,.$$

Then one can rewrite $\Delta = \frac{1}{(\det g)^{1/2}} \partial_j g^{jk} (\det g)^{1/2} \partial_k$ and we come the eigenvalue problem

$$r^{-2}\left[h^2\frac{\partial^2}{\partial h^2} + 3\frac{\partial}{\partial h} + 4\varepsilon h^{-2}\frac{\partial^2}{\partial \bar{z}\partial z} - r^2\frac{\partial^2}{\partial r^2} - 3r\frac{\partial}{\partial r}\right] f_\nu(\bar{z},h,z;r) = \nu^2 f_\nu(\bar{z},h,z;r)\,. \quad (7)$$

Let $\mathcal{Z}_\nu(x)$ be a cylindric function. We will prove the non-commutative analog of the following statement

**Proposition 1** *The basic harmonics of the eigen-value problem (7) are*

$$f_\nu(\bar{z},h,z;r) = r^{-1}h^{-1}\exp(i\mu z + i\bar{\mu}\bar{z}) \times \mathcal{Z}_\alpha(r\nu)\mathcal{Z}_\alpha(2i\varepsilon^{1/2}|\mu|h^{-1})\,, \varepsilon = \pm 1\,, \quad (8)$$

*and*

$$f_\nu(\bar{z},h,z;r) = h^{\alpha-1}\exp(i\mu z + i\bar{\mu}\bar{z})\,,\quad \varepsilon = 0\,,\quad \alpha^2 = \nu^2 + 1\,,$$

*where* $\mu, \alpha \in \mathbb{C}$.

It follows from (7) that the restrictions of the Klein-Gordon equation to the homogeneous spaces assume the form

$$\left(h^2\frac{\partial^2}{\partial h^2} + 3\frac{\partial}{\partial h} + 4\varepsilon h^{-2}\frac{\partial^2}{\partial \bar{z}\partial z}\right) f_\nu(h,z,\bar{z}) = (\nu^2 - 1) f_\nu(h,z,\bar{z})\,,$$

$$\mathbf{L} \to \varepsilon = 1\,,\quad \mathbf{IL} \to \varepsilon = -1\,.$$

Thus, we come to the following statement

**Corollary 1** *The basic harmonics on* **L**, **IL** *and the light-cone* $\mathbf{C}^{1,3}$ *are*

$$f_\nu(\bar{z},h,z) = h^{-1}\exp(i\mu z + i\bar{\mu}\bar{z})\mathcal{Z}_{\nu^2-1}(2i\varepsilon^{\frac{1}{2}}|\mu|h^{-1})\,, \varepsilon = \pm 1\,, \quad (9)$$

*and*

$$f_\nu(\bar{z},h,z) = h^{\alpha-1}\exp(i\mu z + i\bar{\mu}\bar{z})\,,\quad \varepsilon = 0\,,\quad \alpha^2 = \nu^2 + 1\,. \quad (10)$$

# 3 Non-commutative 4d Minkowski space $\mathbb{M}^{1,3}_{\delta,q}$

## 3.1 Definition

We define an algebra generated by matrix elements of (2).

**Definition 1** *The non-commutative 4d Minkowski space* $\mathbb{M}^{1,3}_{\delta,q}$, $0 < q \leq 1$, $\delta \in \mathbb{N}$ *is the unital associative algebra with the anti-involution* $*$ *and four generators* $X_j$, $j = 1, \dots, 4$ *with the quadratic relations*

$$\begin{gathered} X_1X_3 = q^{-\delta}X_3X_1\,, \quad X_1X_2 = q^{\delta}X_2X_1\,, \\ [X_2, X_3] = q^{\delta-2}(1-q^2)X_1X_4\,, \\ X_2X_4 = q^{\delta-2}X_4X_2\,, \quad X_3X_4 = q^{-\delta+2}X_4X_3\,, \\ [X_1, X_4] = 0\,, X_1^* = X_1, \ X_2^* = X_3, \ X_4^* = X_4\,. \end{gathered} \tag{11}$$

This space was described in [3]. Following this approach we cast the relations in $\mathbb{M}^{1,3}_{\delta,q}$ in the form of the reflection equation. Consider the basis in Mat(2)

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Define two $R$-matrices

$$\begin{aligned} R(q) &= q^{-1}(E_1 \otimes E_1 + E_4 \otimes E_4) + (E_1 \otimes E_4 + E_4 \otimes E_1) + q^{-1}(1-q^2)E_3 \otimes E_2\,, \\ R^{(2)}(q) &= (E_1 \otimes E_1 + E_4 \otimes E_4) + q^{\delta-1}(E_1 \otimes E_4 + E_4 \otimes E_1)\,. \end{aligned}$$

It can be checked straightforwardly that the relations (11) are equivalent to the reflection equation

$$R(q)\mathbf{X}^{(1)}R^{(2)}(q)\mathbf{X}^{(2)} = \mathbf{X}^{(2)}R^{(2)}(q)\mathbf{X}^{(1)}R^{\dagger}(q)\,,$$

where $\mathbf{X}^{(1)} = \mathbf{X} \otimes Id$ and $\mathbf{X}^{(2)} = Id \otimes \mathbf{X}$ and

$$\begin{aligned} R^{\dagger}(q) &= q^{-1}(E_1 \otimes E_1 + E_4 \otimes E_4) \\ &+ (E_1 \otimes E_4 + E_4 \otimes E_1) + q^{-1}(1-q^2)E_2 \otimes E_3\,. \end{aligned}$$

The algebra $\mathbb{M}^{1,3}_{\delta,q}$ has two independent Casimir elements

$$K_1 = X_1^{\delta-2}X_4^{\delta}\,, \quad K_2 = X_1X_4 - q^{-\delta}X_3X_2\,. \tag{12}$$

The Casimir operator $K_2$, see (12), is the quantum determinant $K_2 = \det_q \mathbf{X}$. In an irreducible module over $\mathbb{M}^{1,3}_{\delta,q}$ this operator is a scalar: $K_2 = \varepsilon r^2 \in \mathbb{R}$. It allows us to define the time-like part $\mathbb{M}^{1,3+}_{\delta,q}$, ($\varepsilon = 1$), the space-like part $\mathbb{M}^{4-}_{\delta,q}$, ($\varepsilon = -1$), and the light cone $\mathbf{C}_{\delta,q}$, ($\varepsilon = 0$).

### 3.2 Quantum Lorentz group action on $\mathbb{M}^{1,3}_{\delta,q}$

We start with a pair of the standard $\mathcal{U}_q(\mathrm{SL}_2)$ Hopf algebra [4]. The first one is generated by $A, B, C, D$ and the unit 1 with relations

$$AD = DA = 1,\ AB = qBA,\ BD = qDB,$$
$$AC = q^{-1}CA,\ CD = q^{-1}DC, \tag{13}$$
$$[B, C] = \frac{1}{q - q^{-1}}(A^2 - D^2).$$

There is a copy of this algebra $\mathcal{U}^*_q(\mathrm{SL}_2)$ generated by $A^*, B^*, C^*, D^*$ with the relations coming from (13) $U^*V^* = (VU)^*$. They commute with $A, B, C, D$. The pair $\mathcal{U}_q(\mathrm{SL}_2)$, $\mathcal{U}^*_q(\mathrm{SL}_2)$ forms a Hopf algebra $\mathcal{U}^{(s)}_q(\mathrm{SL}_2)$, where the coproduct and the antipode are twisted in the consistent way

$$\Delta(A) = A \otimes A,$$
$$\Delta(B) = A \otimes B + B \otimes D(A^*)^s, \tag{14}$$
$$\Delta(C) = A \otimes C + C \otimes D(A^*)^{-s}.$$

$$S\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}(A^*)^{-s}B \\ -q(A^*)^sC & A \end{pmatrix}.$$

The counit on $U^{(s)}_q(\mathrm{SL}(2, \mathbb{C}))$ assumes the form

$$\varepsilon(A) = 1,\quad \varepsilon(B, C) = 0\,. \tag{15}$$

There is the Casimir element in $U^{(s)}_q(\mathrm{SL}(2, \mathbb{C}))$ which commutes with any $u \in U_q(SL_2(\mathbf{C}))$:

$$\Omega_q := \frac{(q^{-1} + q)(A^2 + A^{-2}) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(BC + CB). \tag{16}$$

$\mathbb{M}^{1,3}_{\delta,q}$ it is a right module over the Hopf algebra $\mathcal{U}^{(s)}_q(\mathrm{SL}(2, \mathbb{C}))$.
We define the action of the quantum group $\mathcal{U}^{(s)}_q(\mathrm{SL}(2, \mathbb{C}))$ on $\mathbb{M}^{1,3}_{\delta,q}$:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.A = \begin{pmatrix} q^{\frac{1}{2}}X_1 & q^{-\frac{1}{2}}X_2 \\ q^{\frac{1}{2}}X_3 & q^{-\frac{1}{2}}X_4 \end{pmatrix},$$

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.B = \begin{pmatrix} 0 & X_1 \\ 0 & X_3 \end{pmatrix},$$

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.C = \begin{pmatrix} X_2 & 0 \\ X_4 & 0 \end{pmatrix},$$

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.A^* = \begin{pmatrix} q^{\frac{1-\delta}{s}}X_1 & q^{\frac{1-\delta}{s}}X_2 \\ q^{\frac{\delta-1}{s}}X_3 & q^{\frac{\delta-1}{s}}X_4 \end{pmatrix}. \tag{17}$$

The direct calculations show that the commutation relations in $\mathbb{M}^{1,3}_{\delta,q}$ are compatible with the coproduct in $\mathcal{U}^{(s)}_q(\mathrm{SL}(2, \mathbb{C}))$.

Similarly, one can define the left action of $\mathcal{U}_q^{(s)}(\mathrm{SL}(2,\mathbb{C}))$ on $\mathbb{M}^{1,3}_{\delta,q}$. Let

$$w(m,k,l,n) = X_3^m X_1^k X_4^l X_2^n$$

be the ordered monomial. Define the Schwartz space $\mathcal{S}(\mathbb{M}^{1,3}_{\delta,q})$ as the series with the rapidly decreasing coefficients

$$\mathcal{S}(\mathbb{M}^{1,3}) = \{\ddagger f(X_3,X_1,X_4,X_2)\ddagger = \sum_{m,k,l,n} a_{m,k,l,n} w(m,k,l,n)\,,\ a_{m,k,l,n} \in \mathbb{C}\}\,, \tag{18}$$

with

$$|a_{m,k,l,n}| < (1+m^2+k^2+l^2+n^2)^j,$$

for any $j \in \mathbb{N}$, when $|m|,|k|,|l|,|n| \to \infty$.

**Proposition 2** *The Jackson integral*

$$\langle f\rangle = \int d_{q^2}X_3 d_{q^2}X_1 d_{q^2}X_4 d_{q^2}X_2 \ddagger f(X_1,X_2,X_3,X_4)\ddagger \tag{19}$$

*is an invariant functional on $\mathcal{S}(\mathbb{M}^{1,3}_{\delta,q})$ with respect to the following action of $\mathcal{U}_q^{(s)}(\mathrm{SL}(2,\mathbb{C}))$: $\langle f.u\rangle = \varepsilon(u)\langle f\rangle$ , where $\varepsilon(u)$ is the counit (15).*

# 4 Horospheric description

## 4.1 Horospheric generators

We introduce another set of generators – the non-commutative analog of the horospheric coordinates $(Z^*, H, Z, R)$, $(H^* = H,\ R^* = R,\ (Z^*)^* = Z)$

$$X_1 = RH\,, \quad X_2 = RHZ\,, \quad X_3 = RZ^*H\,, \tag{20}$$

$$X_4 = R(Z^*HZ + \varepsilon H^{-1})\,, \quad \varepsilon = \pm 1,\ 0\,. \tag{21}$$

The defining relations

$$\begin{gathered} ZH = q^{-\delta}HZ\,, \quad Z^*H = q^{\delta}HZ^*\,, \\ [R,H] = [R,Z] = [R,Z^*] = 0\,, \\ ZZ^* = q^{2\delta-2}Z^*Z - \varepsilon q^{\delta-2}(1-q^2)H^{-2}\,. \end{gathered} \tag{22}$$

yield the relations (11). The Casimir elements are

$$K_2 = \varepsilon R^2\,, \ K_1 = R^{2\delta-2}H^{\delta-2}(Z^*HZ + \varepsilon H^{-1})^{\delta}\,. \tag{23}$$

The inverse relations assume the form

$$\begin{gathered} H = R^{-1}X_1\,, \quad Z = X_1^{-1}X_2\,, \quad Z^* = X_3X_1^{-1}\,, \\ R = \varepsilon K_1\,. \end{gathered}$$

In terms of the horospheric generators the action of $\mathcal{U}_q^{(s)}(\mathrm{SL}(2,\mathbb{C}))$ takes the form

$$\begin{array}{lll} Z^*.A = z^*\,, & H.A = q^{\frac{1}{2}}H\,, & Z.A = q^{-1}z\,, \\ Z^*.A^* = q^{\frac{2\delta-2}{s}}Z^*\,, & H.A^* = q^{\frac{\delta-1}{s}}H\,, & Z.A^* = Z\,, \\ Z^*.B = 0\,, & H.B = 0\,, & Z.B = q^{-\frac{1}{2}}\,, \\ Z^*.C = q^{\frac{3}{2}-\delta}H^{-2}\,, & H.C = HZ\,, & Z.C = -q^{\frac{1}{2}}Z^2\,, \end{array}$$

$$R.A = R,\ R.A^* = R,\ R.B = 0,\ R.C = 0\,. \tag{24}$$

It follows from these relations that $R$ is invariant with respect to the $\mathcal{U}_q^{(s)}(\mathrm{SL}(2,\mathbb{C}))$ action $R.u = \varepsilon(u)R$.

Define the analog of the Schwartz space $\mathcal{S}(\mathbb{M}^{1,3}_{\delta,q})$, see (18), in terms of the ordered monomial $\hat{w}(m,k,n) = Z^{*m}H^kZ^n$. Since $R$ is a center element its position is irrelevant. Let

$$\ddagger f(Z^*,H,Z,R)\ddagger = \sum_{m,k,n,l} a_{m,k,n,l}\hat{w}(m,k,n)R^l\,, \ a_{m,k,n,l} \in \mathbb{C}\,. \tag{25}$$

For $\mathcal{S}(\mathbb{M}^{1,3}_{\delta,q})$ the coefficients satisfy the condition

$$|a_{m,k,n,l}| < (1 + m^2 + k^2 + l^2 + n^2)^j,$$

for any $j \in \mathbb{N}$, when $|m|, |k|, |l|, |n| \to \infty$.

The invariant integral (19) is well defined functional on (25). It assumes the form

$$I_{q^2}(f) = \int d_{q^2}Z^* d_{q^2}H d_{q^2}Z d_{q^2}R \ddagger f(Z^*,H,Z,R)H\ddagger\,.$$

### 4.2 Homogeneous spaces

Consider an irreducible representation of algebra (22). Then one can fix the Casimir operator (23) $K_2 = \varepsilon R^2$, $R^2 = r^2 \in \mathbb{R}^+$. It allows us to define the non-commutative analog of Lobachevsky spaces and the cone. Let us fix the ideal $I_\varepsilon = \{K_2 - \varepsilon r^2 = 0\}$. Then

$$\begin{aligned} \mathcal{S}(\mathbb{M}^{1,3}_{\delta,q})/I_\varepsilon &\sim \mathbf{L}\ \ (\varepsilon = 1)\,, \\ &\sim \mathbf{IL}\ \ (\varepsilon = -1)\,, \\ &\sim \mathbf{C}^{1,3}_{\delta,q}\ \ (\varepsilon = 0)\,. \end{aligned}$$

As we observed above the action of the quantum Lorentz group preserves these spaces. It justifies the notion of homogeneous spaces in the noncommutative situation.

We can directly define their generators using the horospheric description of $\mathbb{M}^{1,3}_{\delta,q}$.

**Definition 2** *The non-commutative Lobachevsky space $\mathbf{L}_{\delta,q}$ ($H_3$), the non-commutative Imaginary Lobachevsky space $\mathbf{IL}_{\delta,\mathbf{q}}$ ($dS_3$) and the non-commutative cone $\mathbf{C}^{1,3}_{q,\delta}$ are the associative unital algebras with an anti-involution and the defining relations*

$$ZH = q^{-\delta}HZ\,, \quad Z^*H = q^{\delta}HZ^*\,,$$
$$ZZ^* = q^{2\delta-2}Z^*Z - \varepsilon q^{\delta-2}(1-q^2)H^{-2}\,.$$
$$(Z)^* = Z^*\,, \quad H^* = H\,,$$
$$\mathrm{H}_3 \sim \varepsilon = 1\,, \quad \mathrm{dS}_3 \sim \varepsilon = -1\,, \quad \mathbf{C}^{1,3}_{\delta,q} \sim \varepsilon = 0\,.$$

In addition we define the non-commutative absolute.

**Definition 3** *The non-commutative absolute $\Xi_{\delta,q}$ is the associative algebra with two generators and the commutation relation*

$$ZZ^* = q^{-2+2\delta}Z^*Z\,. \tag{26}$$

## 5 The Laplace operator and its eigen-functions

### 5.1 The Laplace operator on $\mathbb{M}^{1,3}_{\delta,q}$

Consider the action of $\mathcal{U}_q^{(s)}(\mathrm{SL}(2,\mathbb{C}))$ on the ordered monomials $w(m,k,l,n) = X_2^m X_1^k X_4^l X_2^n$. It follows from (17) that

$$\begin{aligned} w(m,k,l,n).A &= q^{\frac{m+k-l-n}{2}} w(m,k,l,n)\,, \\ w(m,k,l,n).A^* &= q^{\frac{(\delta-1)(m-k+l-n)}{s}} w(m,k,l,n)\,, \\ w(m,k,l,n).B &= \\ & q^{\frac{m+k-l-n+1}{2}-\delta(n-1)}\frac{1-q^{2n}}{1-q^2}w(m,k+1,l,n-1) + \\ & q^{\frac{m+k-5l+3n+5}{2}-\delta(k-l+n+1)}\frac{1-q^{2l}}{1-q^2}w(m+1,k,l-1,n)\,, \\ w(m,k,l,n).C &= q^{\frac{m-3k-l-n+3}{2}+\delta n}\frac{1-q^{2k}}{1-q^2}w(m,k-1,l,n+1) \\ &+ q^{\frac{-3m-3k+3l-n+3}{2}+\delta(k-l+n)}\frac{1-q^{2m}}{1-q^2}w(m-1,k,l+1,n)\,. \end{aligned} \tag{27}$$

Introduce the group-like operator $M$ that acts on $\mathbb{M}^{1,3}_{\delta,q}$ as

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.M = \begin{pmatrix} q^{\frac{1}{2}}X_1 & q^{\frac{1}{2}}X_2 \\ q^{\frac{1}{2}}X_3 & q^{\frac{1}{2}}X_4 \end{pmatrix}. \tag{28}$$

It has the following properties

$$M^* = M, \quad \Delta(M) = M \otimes M, \quad \epsilon(M) = 1,$$
$$S(M) = M^{-1}.$$

Evidently, $M$ commutes with $A, B, C$ and $A^*$. Define the Hopf algebra $\mathcal{U}_q(\mathrm{GL}(2,\mathbb{C}))$ generated by $A, B, C$ and $M$. It is the quantum deformation of the classical algebra $\mathrm{GL}(2,\mathbb{C})$. Let

$$\Omega_{q,M} := \frac{(q^{-\frac{1}{2}}M^{-1} - q^{\frac{1}{2}}M)^2}{(q^{-1} - q)^2}. \tag{29}$$

Consider the following Casimir element of $\mathcal{U}_q(\mathrm{GL}(2,\mathbb{C}))$

$$\Delta_q = [\Omega_q - \Omega_{q,M}]R^{-2}. \tag{30}$$

This operator is the quantum analog of the Laplace operator $\Delta$, see (7). Define the partial differentiation acting on $w(m,k,l,n)$ in such a way that it does not break the ordering. It means, in particular, that the differentiation of the ordered monomial with respect, for example, $X_1$ takes the form

$$D_{X_1}w(m,k,l,n) = \frac{1-q^{2k}}{1-q^2}w(m,k-1,l,n).$$

Let $T_X f(X) = f(qX)$.

**Proposition 3** *The action of the Laplace operator on* $\mathbb{M}^{1,3}_{\delta,q}$ *assumes the form*

$$\begin{aligned} &f(X_3,X_1,X_4,X_2).\Delta_q = \\ &\Big\{\frac{1}{(q-q^{-1})^2}\Big[q^{-1}(T^{-1}_{X_1}T^{-1}_{X_2}T_{X_3}T^{-1}_{X_4} - T^{-1}_{X_1}T_{X_2}T_{X_3}T^{-1}_{X_4} \\ &+T^{-1}_{X_1}T_{X_2}T^{-1}_{X_3}T^{-1}_{X_4} - T^{-1}_{X_1}T^{-1}_{X_2}T^{-1}_{X_3}T^{-1}_{X_4}) - q(T^{-1}_{X_1}T_{X_2}T_{X_3}T_{X_4} + T_{X_1}T_{X_2}T_{X_3}T_{X_4} \\ &+T_{X_1}T_{X_2}T_{X_3}T^{-1}_{X_4} - T^{-1}_{X_1}T_{X_2}T_{X_3}T^{-1}_{X_4})\Big] + q^{1+\delta}T^{-k(-\delta+1)}_{X_1}T^{-1}_{X_2}T^{-1}_{X_3}T^{l(1-\delta)}_{X_4}D_{X_2}D_{X_3} \\ &+q^{5-\delta}T^{-k(\delta+1)}_{X_1}T_{X_2}T_{X_3}T^{-l(3-\delta)}_{X_4}D_{X_1}D_{X_4}\Big\} f(X_3,X_1,X_4,X_2). \end{aligned} \tag{31}$$

**Remark 1** *In the classical limit* $\lim_{q\to 1}\Delta_q = \Delta$, *see (7).*

## 5.2 The Laplace operator on $\mathbb{M}^{1,3}_{\delta.q}$ in terms of the horospheric generators

Define the ordered monomial

$$\hat{w}(m,k,n) = (Z^*)^m H^k Z^n,$$

and let

$$F(Z^*,H,Z) = \sum_{m,k,n} a_{m,k,n}\hat{w}(m,k,n).$$

Consider the action of the operator $\Delta_q$ on the Schwartz space (18).

**Proposition 4** *The action of the Casimir operator $\Delta_q$ in terms of horospheric generators takes the form*

$$\begin{aligned} & F(Z^*, H, Z, R).\Delta_q = \\ & \frac{1}{(1-q^2)^2}\left[q^{-1}T_H - q^2 + qT_H^{-1}\right] F(Z^*, H, Z, R) \\ & +\varepsilon q^{1-\delta} D_{Z^*} D_Z \ddagger H^{-2} T_H^{\delta-1} F(Z^*, H, Z, R) \ddagger . \end{aligned} \tag{32}$$

**Remark 2** *In the classical limit (32) takes the form of the Laplace operator in horospheric coordinates* $\lim_{q\to 1}\Delta_q = \Delta$, *see (7).*

Our main goal is to find the eigen-functions of $\Delta_q$

$$F_\nu(Z^*, H, Z, R).\Delta_q = \left[\frac{\nu}{2}\right]_{q^2}^2 F_\nu(Z^*, H, Z, R) . \tag{33}$$

These functions are expressed through the $q$-exponents and the three types of $q$-cylindric functions. For $|q| \neq 1$ they can be defined by the expansion

$$\mathbf{Z}_\alpha^{(j)}(z) = \frac{1}{(1-q^2)^\alpha \Gamma_{q^2}(\alpha+1)} \sum_{m=0}^{\infty} \frac{q^{(2-\delta)m(m+\alpha)} z^{\alpha+2m}}{(q^2, q^2)_m (q^{2\alpha+2}, q^2)_m 2^{\alpha+2m}} , \tag{34}$$

$$j = -\frac{3}{2}\delta^2 + \frac{5}{2}\delta + 2 ,$$

where $\Gamma_{q^2}(\alpha+1)$ is the $q^2$-$\Gamma$-function. We assume that

$$\frac{|z|}{2(1-q^2)} < 1 \quad \text{for} \quad \delta = 2.$$

The non-commutative analog of the horospheric elementary harmonics (8) has the following form

**Proposition 5** *The basic solutions of (33) are defined as*

$$F_\nu(Z^*, H, Z, R) = \mathbf{e}(\bar{\mu} Z^*) V_\alpha(H) \mathbf{e}(\mu Z) \Xi_{\nu,\alpha}(R) , \quad (\varepsilon \neq 0) ,$$

*where* $\mu, \alpha \in \mathbb{C}$,

$$\begin{aligned} V_\alpha(H) &= H^{-1}\mathbf{Z}_\alpha^{(j)}(2(-\varepsilon)^{1/2}|\mu| q^{-\delta/2} H^{-1}) , \\ \Xi_{\nu,\alpha}(R) &= \frac{1}{R}\mathbf{Z}_\alpha^{(3)}(2q^{1-\nu/2}\frac{1-q^\nu}{1-q^2} R) . \end{aligned}$$

Represent the solutions in the form

$$F_\nu(Z^*, H, Z, R) = V_\alpha(Z^*, H, Z)\Xi_{\nu,\alpha}(R) .$$

Substituting it in (33) and using the comultiplication relations (14), we find

$$\begin{aligned} & (V_\alpha(Z^*, H, Z).\Omega_q)\,(\Xi_{\nu,\alpha}(R).\Omega_q)\, R^{-2} \\ & - (V_\alpha(Z^*, H, Z).\Omega_{q,M})\,(\Xi_{\nu,\alpha}(R).\Omega_{q,M})\, R^{-2} \\ & - \left[\frac{\nu}{2}\right]_{q^2}^2 V_\alpha(Z^*, H, Z)\Xi_{\nu,\alpha}(R) = 0 . \end{aligned}$$

It follows from (24) that it can be rewritten as

$$\begin{aligned}&(V_\alpha(Z^*,H,Z).\Omega_q)\,\Xi_{\nu,\alpha}(R)R^{-2}\\&-V_\alpha(Z^*,H,Z)\left(\Xi_{\nu,\alpha}(R).\Omega_{q,M}R^{-2}\right)\\&-\left[\frac{\nu}{2}\right]^2_{q^2}V_\alpha(Z^*,H,Z)\Xi_{\nu,\alpha}(R)=0\,.\end{aligned}$$

In this way we come to the equations

$$V_\alpha(Z^*,H,Z).\Omega_q-\frac{q^{-\alpha+2}-2q^2+q^{\alpha+2}}{(1-q^2)^2}V_\alpha(Z^*,H,Z)=0\,,\tag{35}$$

and

$$\Xi_{\nu,\alpha}(R).\Omega_{q,M}+\Xi_{\nu,\alpha}(R)\left(\frac{q^{-\nu+2}-2q^2+q^{\nu+2}}{(1-q^2)^2}R^2-\frac{q^{-\alpha+2}-2q^2+q^{\alpha+2}}{(1-q^2)^2}\right)=0\tag{36}$$

From (28) and (29) one rewrites the equation (36) as

$$q\Xi_{\nu,\alpha}(q^{-1}R)-(q^{\alpha+2}+q^{-\alpha+2})\Xi_{\nu,\alpha}(R)+q^3\Xi_{\nu,\alpha}(qR)=q^{3-\nu}(1-q^\nu)^2\Xi_{\nu,\alpha}(R)\,.$$

Put

$$z=2q^{-\nu/2}(1-q^\nu)(1-q^2)^{-1}R.$$

Then we come to the difference equation for $q$-Bessel functions with $\delta=1$ and $\mathbf{Z}^{(3)}_\alpha(z)=\Xi_{\nu,\alpha}(R)R$.

Consider now (35) and put

$$\tilde{V}_\alpha(Z^*,H,Z)=\mathbf{e}(\bar\mu Z^*)V_\alpha(H)\mathbf{e}(\mu Z)\,.\tag{37}$$

Assume that $\varepsilon=\pm1$ and

$$V_\alpha(H)=\sum_{k=0}^\infty c_k\frac{(1-q^2)^{2k-2}}{(q^2,q^2)_k(q^{2\alpha+2},q^2)_k}H^{-\alpha-2k-1}\,.$$

Substituting this expression in (37), we express $\tilde{V}_\alpha$ in terms of monomials $\hat{w}(m,k,n)$. Using the action of $\Omega_q$ on monomials we obtain

$$\begin{aligned}&\mathbf{e}(\bar\mu Z^*)\sum_{k=0}^\infty c_k\frac{(1-q^2)^{2k-2}}{(q^2,q^2)_k(q^{2\alpha+2},q^2)_k}q^{-\alpha-2k+2}(1-q^2)(1-q^{2\alpha+2k})H^{-\alpha-2k-1}\mathbf{e}(\mu Z)\\&-\ \varepsilon|\mu|^2\mathbf{e}(\bar\mu Z^*)\sum_{k=0}^\infty c_k\frac{(1-q^2)^{2k}}{(q^2,q^2)_k(q^{2\alpha+2},q^2)_k}q^{(\alpha+2k+2)(\delta-1)}H^{-\alpha-2k-3}\mathbf{e}(\mu Z)=0\,.\end{aligned}$$

Then the coefficients $c_k$ satisfy the recurrence relation

$$\begin{aligned}c_{k+1}&=-\varepsilon\bar\mu\mu c_kq^{2\alpha+4k+2-\delta(\alpha+2k+2)}\,.\\c_k&=(-\varepsilon)^k|\mu|^{2k}q^{(2-\delta)k(k+\alpha)-\delta k}\,.\end{aligned}$$

Then

$$\mathcal{H}_\alpha(H)=q^{\frac{\delta\alpha}{2}}|\mu|^{-\alpha}\sum_{k=0}^\infty(-\varepsilon)^k\frac{q^{(2-\delta)k(k+\alpha)}(1-q^2)^{2k}}{(q^2,q^2)_k(q^{2\alpha+2},q^2)_k}q^{-\frac{\delta}{2}(\alpha+2k)}(\bar\mu\mu)^{\frac{\alpha}{2}+k}H^{-\alpha-2k-1}\,.$$

These series coincide with (34) up to a constant multiplier after replacing $2z$ by $(-\varepsilon)^{1/2}q^{-\delta/2}H^{-1}$.

**Remark 3** *In the classical limit we come to Proposition 2.1*

$$\lim_{q\to 1} \mathbf{e}(\bar{\mu}Z^*)V_\alpha(H)\mathbf{e}(\mu Z)\Xi_{\nu,\alpha}(R) = \exp(i\mu z + i\bar{\mu}\bar{z})v_\alpha(h)\chi_{\nu,\alpha}(r)\,.$$

As in the classical situation one can restrict the operator $\Delta_q$ on the non-commutative homogeneous spaces.

**Corollary 2** *The restrictions of $\Delta_q$ assume the form*

$$\mathbf{L}_{q,\delta}\,, \quad : \quad \Delta_q = \frac{1}{(1-q^2)^2}[q^3T_H - 2q^2 + qT_H^{-1}] + q^{1-\delta}D_{Z^*}D_Z H^{-2}T_H^{\delta-1}\,,$$

$$\mathbf{IL}_{q,\delta}\,, \quad : \quad \Delta_q = \frac{1}{(1-q^2)^2}[q^3T_H - 2q^2 + qT_H^{-1}] - q^{1-\delta}D_{Z^*}D_Z H^{-2}T_H^{\delta-1}\,.$$

Then we obtain the non-commutative deformations of the classical formulas (9), (10).

**Corollary 3** *The basic harmonics on the non-commutative* **L**, **IL** *and the light-cone* $\mathbf{C}^{1,3}_{q,\delta}$ *are*

$$F_\nu(\bar{z}, h, z) = \mathbf{e}(\mu Z^*)H^{-1}\mathbf{Z}^{(j)}_\alpha(2i\varepsilon^{\frac{1}{2}}|\mu|H^{-1})\mathbf{e}(\mu Z)\,, \quad \varepsilon = \pm 1\,,$$

*and*

$$F_\nu(\bar{z}, h, z) = \mathbf{e}(\mu Z^*)H^{\alpha-1}\mathbf{e}(\mu Z)\,, \quad \varepsilon = 0\,.$$

*Here* $\nu^2 = \alpha^2 - 1$.

# References


[1] M. R. Douglas, N. A. Nekrasov, *Rev. Mod. Physics*, **73** (2001), 977, [hep-th/0106048]

[2] R. J. Szabo, *Phys. Rep.*, **378** (2003), 207, [hep-th/0109162]

[3] J. A. de Azcarraga, F. Rodenas, "Deformed Minkowski Spasecs: Classification and Properties", *J. Phys.* **A29** (1996), 1215–1226, [q-alg/9510011]

[4] P. Kulish, N. Reshetikhin, "Quantum linear problem for the sin-Gordon equation and the higher representations", *Zapiski LOMI*, **101** (1981), 101–110